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Replaying neutrino bremsstrahlung with general dispersion relations

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ABSTRACT: It is generally held that neutrinos with superluminal velocity will lose their energy spontaneously by radiating electron-positron pairs, similar to bremsstrahlung process. Recently, this process was closely studied for neutrinos whose energy is roughly proportional to their momentum. Confronted with an increasing amount of superluminal neutrino models, it is urgent to calculate the same process for general dispersion relations. The calculation is performed in this paper, without resorting to any nontrivial frame such as the effective “rest frame”.

KEYWORDS: Lorentz violation, neutrino, special relativity.

Contents

1	Introduction	1
2	Basic assumptions and notation conventions	2
3	Energy threshold	3
4	Decay width	5
5	Examples	9
5.1	Muon decay	10
5.2	Cohen-Glashow model	10
5.3	Mass-dependent Lorentz violation	10
5.4	Velocity of step form	11
5.5	Horava-Lifshitz model	12
6	Conclusion	13

1 Introduction

The OPERA experiment stirred the physics community recently with its astonishing result that neutrinos in this experiment travel apparently faster than light at a high confidence level [1]. If not attributed to systematic errors in the measurement, this result would imply the violation of special relativity. Subsequently, it has inspired a lot of speculation.¹

However, as quickly claimed in [21, 22], several high-energy processes disfavor the superluminal interpretation of the OPERA data. An outstanding example is the bremsstrahlung-like process

$$\nu_\mu \rightarrow \nu_\mu + e^+ + e^-, \quad (1.1)$$

where electron-positron pairs are radiated and hence neutrinos lose their energy efficiently. To explicitly demonstrate this point, the authors of [21] assumed a special dispersion relation roughly of the form $E = v_\nu p$, where v_ν is a constant greater than light velocity. A similar assumption was also taken in [22].

¹As a partial list, see [2–20] and references therein for speculation on this issue from various aspects.

On the other hand, a dispersion relation of the form $E = v_\nu p$ is too oversimplified to accommodate more observational data of neutrino velocity, as summarized and analyzed in [23]. Therefore, it is urgent to extend the calculation of [21, 22] to general dispersion relations. The present paper is devoted to such a calculation. Our results would help to rule out more phenomenological models and hunt for viable models.

In the absence of Lorentz invariance, the calculation is complicated even for the special dispersion relation $E = v_\nu p$, so an effective “mass” was assigned to neutrinos and a “rest frame” was employed in [21]. Our calculation does not resort to such a nontrivial reference frame. Or more explicitly, one may interpret our work as a direct calculation in the laboratory frame. Applied to the above dispersion relation, our result provides a crosscheck for the results in [21, 22].

The paper is organized as follows. Section 2 collects the basic assumptions and conventions of notation in this work. Kinematically there is a threshold energy for the process (1.1). We discuss the dependence of this threshold on dispersion relations in section 3. Section 4 is the main part of our paper, where we calculate the “decay width” of superluminal neutrino via (1.1). Details and techniques are presented clearly. To check and apply our general results in sections 3 and 4, we work out some specific examples in section 5 with given dispersion relations. The results are consistent with [21] quantitatively and [26] qualitatively. Section 6 concludes this paper.

2 Basic assumptions and notation conventions

Throughout the paper, we assume

1. The ordinary conservation law for energy and momentum is intact. In other words, the time and space translations are exact symmetries in the working frame. A case study for violating this assumption can be found in [24].
2. The space is Euclidean and isotropic. Thereby we can work in a spherical coordinate system and define the magnitude of momentum as $p = |\vec{p}|$.
3. In the relevant energy range, the dispersion relation of electron and positron is well characterized by $E^2 = p^2 + m_e^2$. This assumption is in accordance with experiments to date.
4. The dispersion relation of neutrino is either $E = E(p)$ or $p = p(E)$. Here $E(p)$ and $p(E)$ are arbitrary devisable functions. They could be non-monotonic and may involve some parameters such as mass, etc.

In section 3, when deriving the threshold energy of process (1.1), we make one more assumption:

- The neutrino’s dispersion relation reduces to $E^2 = p^2 + m_\nu^2$ at very low energies, typically lower than the threshold energy by a factor $\mathcal{O}(10^{-4})$. See details in section 3.

In section 4, this assumption is replaced by two other assumptions:

- The squared amplitude (4.2) is the same as that in standard model. We will comment on possible loopholes of this assumption in section 6. But to alleviate the complexity of our calculation, we should make such an assumption at the moment.
- The masses of electron and positron are neglected. This assumption seems to be reasonable, because when we study the decay width of high-energy neutrino which is practically around or above GeV scale, the phase space should be dominated by high-energy particles in principle.

Let us clarify the conventions of notation by writing (1.1) in the form

$$\nu(p) \rightarrow \nu(p') + e^+(k') + e^-(k), \quad (2.1)$$

where we have specified the notation of momentum for each particle. One must be careful of the notations of momenta. Taking p for instance, sometimes it stands for the four-vector, but sometimes it stands for the magnitude of three-vector \vec{p} . In our equations, the four-vectors appear usually together with a dot, indicating the inner product with Lorentz signature $\text{diag}(+, -, -, -)$. For example, $p \cdot p = E_p^2 - |\vec{p}|^2$ but $p^2 = |\vec{p}|^2$. So there are no confusions if one is careful.

The velocity of light is a constant, so we set it to 1 throughout this paper.

3 Energy threshold

The assumptions made in the previous section simplify the derivation of threshold energy for (1.1) considerably.² From the energy conservation relation

$$E_p = E_{p'} + E_{k'} + E_k, \quad (3.1)$$

²When our work was in preparation, ref. [25] appeared. The topics in this section and [25] overlap partly. For the completeness of our paper, we keep this section in its own form. Ref. [26], starting from different assumptions, also concerns a partly overlapped subject of this paper. It appeared more recently when we were polishing our work.

we observe that to minimize E_p , one should lower $E_{p'}$, $E_{k'}$ and E_k as much as possible. This can be achieved by deleting all of the transverse momentum components, leading to the reduced conservation law of momentum

$$p = p' + k' + k, \quad (3.2)$$

where $p = |\vec{p}|$, $p = |\vec{p}'|$ and so on are magnitude of momenta as our conventions.

Substituting (3.1) and (3.2) into the dispersion relation of $e^+(k')$ and $e^-(k)$, we have

$$(E_p - E_{p'})^2 - 2E_k(E_p - E_{p'}) - (p - p')^2 + 2k(p - p') = 0. \quad (3.3)$$

In light of dispersion relations of $\nu(p)$, $\nu(p')$ and $e^-(k)$, this equation may be understood as an implicit function of E_p , $E_{p'}$ and E_k . Then we can extremize E_p with respect to $E_{p'}$ and E_k , obtaining

$$\begin{aligned} (E_p - E_k) - (p - k) \frac{E_{p'}}{p'} &= 0, \\ (E_p - E_{p'}) - (p - p') \frac{E_k}{k} &= 0, \end{aligned} \quad (3.4)$$

where conditions $\partial E_p / \partial E_{p'} = \partial E_p / \partial E_k = 0$ and dispersion relations $dp' / dE_{p'} = E_{p'} / p'$, $dk / dE_k = E_k / k$ are used.

From eqs. (3.3) and (3.4), it is not hard to get

$$2E_k = E_p - E_{p'}, \quad 2k = p - p', \quad \frac{E_p}{p} = \frac{E_{p'}}{p'} \quad (3.5)$$

and subsequently

$$E_p^2 - 2E_p E_{p'} + m_\nu^2 = p^2 - 2pp' + 4m_e^2, \quad E_{p'} = \frac{m_\nu}{\sqrt{1 - \frac{p^2}{E_p^2}}}. \quad (3.6)$$

As a result, we find the threshold of (1.1) is given by

$$(E_p^2 - p^2)_{\text{thr.}} = (2m_e + m_\nu)^2. \quad (3.7)$$

This result is in accordance with [25]. To the leading order, it is also compatible with [21] where neutrino's dispersion relation is $E_p/p = v_\nu$. This will be shown in subsection 5.2. As a concrete example, in subsection 5.3 we will utilize (3.7) to get the decay threshold for a toy model of mass-dependent Lorentz violation [23].

Note that when writing down (3.4) and (3.6), we have assumed the dispersion relation $E_{p'}^2 = p'^2 + m_\nu^2$ of $\nu(p')$ at low energies. But the dispersion relation of $\nu(p)$ is irrelevant throughout the derivation. From eqs. (3.6) and (3.7) we can see

$(E_{p'}/E_p)_{\text{thr.}} = m_\nu/(2m_e + m_\nu) \sim \mathcal{O}(10^{-4})$. That means we have assumed the dispersion relation $E_{p'}^2 = p'^2 + m_\nu^2$ for neutrinos at an energy lower than the threshold energy by $\mathcal{O}(10^{-4})$. This assumption is natural and consistent with the observational data of neutrino velocity summarized in [23]. Replacing this assumption with other dispersion relations, one may also restart from (3.3) and follow our method to derive the threshold energy.³

4 Decay width

We are interested in the following process: $\nu(p) \rightarrow \nu(p') + e^+(k') + e^-(k)$. Considering the neutral current of this process, we have⁴

$$\sum_{\text{spin}} \mathcal{M}\mathcal{M}^* = 128G_F^2[(p \cdot k')(k \cdot p')(-\frac{1}{2} + \sin^2 \theta)^2 + (p \cdot k)(k' \cdot p') \sin^4 \theta]. \quad (4.1)$$

To calculate the decay width, we will integrate over p' , k and k' , hence we can make use of the symmetry between k and k' to write the squared amplitude as

$$\sum_{\text{spin}} \mathcal{M}\mathcal{M}^* = 128G_F^2(p \cdot k')(k \cdot p') \left[\left(-\frac{1}{2} + \sin^2 \theta_W \right)^2 + \sin^4 \theta_W \right]. \quad (4.2)$$

The decay width is formally given by

$$\begin{aligned} \Gamma &= \frac{1}{2E_p} \int \frac{d^3\vec{p}'}{(2\pi)^3 2E_{p'}} \int \frac{d^3\vec{k}}{(2\pi)^3 2E_k} \int \frac{d^3\vec{k}'}{(2\pi)^3 2E_{k'}} \frac{1}{2} |\mathcal{M}|^2 (2\pi)^4 \delta^4(p - p' - k' - k) \\ &= \frac{8G_F^2}{(2\pi)^5 E_p} \left(\frac{1}{4} - \sin^2 \theta_W + 2 \sin^4 \theta_W \right) \int \frac{d^3\vec{p}'}{E_{p'}} \int \frac{d^3\vec{k}}{E_k} \int \frac{d^3\vec{k}'}{2E_{k'}} (p \cdot k')(k \cdot p') \delta^4(p - p' - k' - k). \end{aligned} \quad (4.3)$$

Here θ_W is the Weinberg angle.

In the main part of this section, we will focus on the calculation of integral by temporarily forgetting the overall coefficient. Because we are interested in high-energy neutrino decay, the masses of electron and positron will be neglected in our calculation.

³This method is valid if E_p has local minima as an implicit function (3.3) of $E_{p'}$ and E_k . The situation will be more complicated if the configuration of (3.3) does not have a local minimum.

⁴In the first version of our manuscript, the squared amplitude is incomplete. Here we corrected this error and included the last term in (4.2). The difference does not affect most of our calculations. It only modifies an overall factor in the decay width. We are grateful to Zhaohuan Yu, Fedor Bezrukov and Evslin Jarah for communications on this point.

Then the integral in (4.3) can be briefly rewritten as

$$\begin{aligned}
\Gamma_{\text{abb.}} &= \int \frac{d^3 \vec{p}'}{E_{p'}} \int \frac{d^3 \vec{k}}{E_k} \int \frac{d^3 \vec{k}'}{2E_{k'}} (p \cdot k') (k \cdot p') \delta^4(p - p' - k' - k) \\
&= \int \frac{d^3 \vec{p}'}{E_{p'}} \int \frac{d^3 \vec{k}}{E_k} \int d^4 k' \delta(k' \cdot k') \big|_{k'^0 > 0} (p \cdot k') (k \cdot p') \delta^4(p - p' - k' - k) \\
&= \int \frac{d^3 \vec{p}'}{E_{p'}} \int \frac{d^3 \vec{k}}{E_k} [p \cdot (p - p' - k)] (k \cdot p') \delta((p - p' - k) \cdot (p - p' - k)) \big|_{E_p - E_{p'} - k > 0}.
\end{aligned} \tag{4.4}$$

Making use of relations $k \cdot k = k' \cdot k' = 0$ and

$$k \cdot p' = k \cdot (p - k - k') = k \cdot p - \frac{(k + k') \cdot (k + k')}{2} = k \cdot p - \frac{(p - p') \cdot (p - p')}{2}, \tag{4.5}$$

we obtain

$$\begin{aligned}
\Gamma_{\text{abb.}} &= \int \frac{d^3 \vec{p}'}{E_{p'}} \int \frac{d^3 \vec{k}}{E_k} [p \cdot (p - p') - p \cdot k] \left[k \cdot p - \frac{(p - p') \cdot (p - p')}{2} \right] \\
&\quad \times \delta((p - p') \cdot (p - p') - 2k \cdot (p - p')) \bigg|_{E_p - E_{p'} - k > 0} \\
&= \int \frac{d^3 \vec{p}'}{E_{p'}} \int k dk \sin \theta_1 d\theta_1 d\varphi_1 \left[p \cdot (p - p') - E_p E_k + \vec{p} \cdot \vec{k} \right] \\
&\quad \times \left[E_p E_k - \vec{p} \cdot \vec{k} - \frac{(p - p') \cdot (p - p')}{2} \right] \\
&\quad \times \delta\left((p - p') \cdot (p - p') - 2E_k(E_p - E_{p'}) + 2\vec{k} \cdot (\vec{p} - \vec{p}')\right) \bigg|_{E_p - E_{p'} - k > 0}. \tag{4.6}
\end{aligned}$$

Here θ_1 is defined as the angle between \vec{k} and $\vec{p} - \vec{p}'$. We will define θ_2 as the angle between \vec{p} and \vec{p}' . The relative directions and angles between the relevant momentum vectors are depicted in figure 1. It is convenient to express the inner product of \vec{p} and \vec{k} in terms of the new coordinates,

$$\vec{p} \cdot \vec{k} = pk \frac{\cos \theta_1 (p - p' \cos \theta_2) + p' \sin \theta_1 \cos \varphi_1 \sin \theta_2}{|\vec{p} - \vec{p}'|}. \tag{4.7}$$

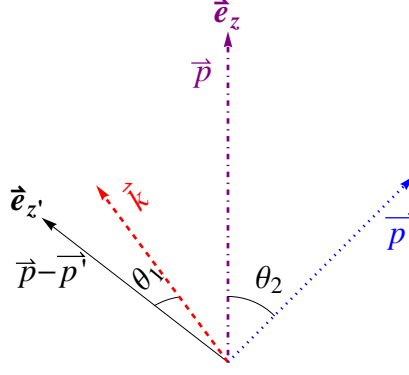


Figure 1. (color online). The relative directions of vectors \vec{p} , \vec{p}' , $\vec{p}-\vec{p}'$ and \vec{k} in three spatial dimensions. The zenith angle θ_1 is defined as the angle between \vec{k} and $\vec{p}-\vec{p}'$, and θ_2 as the angle between \vec{p} and \vec{p}' . As shown in the picture, axes \vec{e}_z and $\vec{e}_{z'}$ coincide with directions of \vec{p} and $\vec{p}-\vec{p}'$ respectively. Projecting \vec{k} on the plane perpendicular to $\vec{e}_{z'}$, we can define one azimuth angle φ_1 . Similarly, the other azimuth angle φ_2 can be defined by projection of \vec{p}' on the plane perpendicular to \vec{e}_z .

In terms of the new coordinates, the integration takes the form

$$\begin{aligned}
\Gamma_{\text{abb.}} &= \int \frac{d^3 \vec{p}'}{E_{p'}} \int k dk \sin \theta_1 d\theta_1 d\varphi_1 \left[p \cdot (p - p') - E_p E_k \right. \\
&\quad \left. + p k \frac{\cos \theta_1 (p - p' \cos \theta_2) + p' \sin \theta_1 \cos \varphi_1 \sin \theta_2}{|\vec{p} - \vec{p}'|} \right] \\
&\quad \times \left[E_p E_k - p k \frac{\cos \theta_1 (p - p' \cos \theta_2) + p' \sin \theta_1 \cos \varphi_1 \sin \theta_2}{|\vec{p} - \vec{p}'|} - \frac{(p - p') \cdot (p - p')}{2} \right] \\
&\quad \times \delta \left((p - p') \cdot (p - p') - 2E_k (E_p - E_{p'}) + 2k |\vec{p} - \vec{p}'| \cos \theta_1 \right) \Big|_{E_p - E_{p'} - k > 0} \\
&= \int \frac{d^3 \vec{p}'}{E_{p'}} \int k dk d\varphi_1 \int_{-1}^1 dx \left[p \cdot (p - p') - E_p E_k \right. \\
&\quad \left. + p k \frac{x(p - p' \cos \theta_2) + p' \sqrt{1 - x^2} \cos \varphi_1 \sin \theta_2}{|\vec{p} - \vec{p}'|} \right] \\
&\quad \times \left[E_p E_k - p k \frac{x(p - p' \cos \theta_2) + p' \sqrt{1 - x^2} \cos \varphi_1 \sin \theta_2}{|\vec{p} - \vec{p}'|} - \frac{(p - p') \cdot (p - p')}{2} \right] \\
&\quad \times \delta \left((p - p') \cdot (p - p') - 2E_k (E_p - E_{p'}) + 2k |\vec{p} - \vec{p}'| x \right) \Big|_{E_p - E_{p'} - k > 0}. \tag{4.8}
\end{aligned}$$

With the newly introduced variable $x = \cos \theta_1 \in [-1, 1]$, we note that the function

$$f(x) = (p - p') \cdot (p - p') - 2k(E_p - E_{p'}) + 2k|\vec{p} - \vec{p}'|x \quad (4.9)$$

has a single root

$$x_0 = \frac{2k(E_p - E_{p'}) - (p - p') \cdot (p - p')}{2k|\vec{p} - \vec{p}'|} \quad (4.10)$$

and its first-order derivative

$$f'(x) = 2k|\vec{p} - \vec{p}'|. \quad (4.11)$$

Therefore we can integrate x out of the delta function and quickly obtain

$$\begin{aligned} \Gamma_{\text{abb.}} &= \int \frac{d^3 \vec{p}'}{E_{p'}} \int k dk d\varphi_1 \left[p \cdot (p - p') - kE_p \right. \\ &\quad \left. + kp \frac{x_0(p - p' \cos \theta_2) + p' \sqrt{1 - x_0^2} \cos \varphi_1 \sin \theta_2}{|\vec{p} - \vec{p}'|} \right] \\ &\quad \times [E_p k - pk \frac{x_0(p - p' \cos \theta_2) + p' \sqrt{1 - x_0^2} \cos \varphi_1 \sin \theta_2}{|\vec{p} - \vec{p}'|} - \frac{(p - p') \cdot (p - p')}{2}] \\ &\quad \times \frac{1}{2k|\vec{p} - \vec{p}'|} \Big|_{E_p - E_{p'} - k > 0} \\ &= \int \frac{d^3 \vec{p}'}{2E_{p'}|\vec{p} - \vec{p}'|} \int dk \left\{ 2\pi \left[p \cdot (p - p') - kE_p + p \frac{kx_0(p - p' \cos \theta_2)}{|\vec{p} - \vec{p}'|} \right] \left[E_p k \right. \right. \\ &\quad \left. \left. - p \frac{kx_0(p - p' \cos \theta_2)}{|\vec{p} - \vec{p}'|} - \frac{(p - p') \cdot (p - p')}{2} \right] - \pi \frac{(pp')^2(k^2 - k^2 x_0^2) \sin^2 \theta_2}{|\vec{p} - \vec{p}'|^2} \right\} \Big|_{E_p - E_{p'} - k > 0} \quad (4.12) \end{aligned}$$

One may check that the integrand of (4.12) is independent of φ_2 . Its dependence on k is quite simple. Its domain of integration is determined by $-1 < x_0 < 1$, $E_p - E_{p'} - k > 0$, or namely

$$\begin{aligned} \frac{(p - p') \cdot (p - p')}{2(E_p - E_{p'}) + 2|\vec{p} - \vec{p}'|} < k < \frac{(p - p') \cdot (p - p')}{2(E_p - E_{p'}) - 2|\vec{p} - \vec{p}'|}, \\ E_p - E_{p'} > |\vec{p} - \vec{p}'|. \end{aligned} \quad (4.13)$$

So we can integrate over variables k and φ_2 straightforwardly. The calculation is a little tedious, yielding

$$\begin{aligned} \Gamma &= \frac{8G_F^2}{(2\pi)^5 E_p} \left(\frac{1}{4} - \sin^2 \theta_W + 2 \sin^4 \theta_W \right) \pi^2 \int \int \frac{p'^2 dp' dy}{6E_{p'}} \left[3E_p E_{p'}^3 \right. \\ &\quad \left. + (-6E_p^2 + 2p^2 - 3pp'y)E_{p'}^2 + (3E_p^3 - 3E_p p^2 - 3E_p p'^2 + 8E_p pp'y)E_{p'} \right. \\ &\quad \left. + 3py p'^3 + (2E_p^2 - 2p^2 - 4p^2 y^2)p'^2 + (3p^3 y - 3pE_p^2 y)p' \right]. \end{aligned} \quad (4.14)$$

The domain of integration is

$$p^2 + p'^2 - 2pp'y < (E_p - E_{p'})^2, \quad (4.15)$$

$$-1 < y < 1. \quad (4.16)$$

In the above, we employed notation $y = \cos \theta_2$ and turned on the overall factor in front of the integral as given by (4.3).

The expression (4.14) for decay width is one of our main results. In section 5, we will apply it to several examples with explicit dispersion relations. Before proceed, let us make some comments on (4.14).

First, we would like to show that the decay width (4.14) is positive-definite, as one should have anticipated from its definition (4.3). For this purpose, it is enough to focus on the integrand inside the square brackets, which can be transformed to

$$\begin{aligned} [\cdots] &= 3(E_p E_{p'} - pp'y) [(E_p - E_{p'})^2 - (p^2 + p'^2 - 2pp'y)] + 2(E_p p' - E_{p'} p)^2 \\ &\quad + 2pp'(1 - y) [2E_p E_{p'} - pp'(1 + y)]. \end{aligned} \quad (4.17)$$

Taking account of inequalities (4.15) and (4.16), this expression is nonnegative if $E_p E_{p'} \geq pp'$. This condition is well-satisfied if $E_p - p \geq 0$, $E_{p'} - p' \geq 0$ at lower energy and $dE_p/dp \geq 1$, $dE_{p'}/dp' \geq 1$ in the energy region of superluminal neutrino.

Second, we note that (4.15) puts a lower limit of integration $y > [p^2 + p'^2 - (E_p - E_{p'})^2]/(2pp')$. In most situations, we have $E_p - E_{p'} \leq p + p'$, then this limit is more stringent than $y > -1$, and hence the domain of integration is simply $[p^2 + p'^2 - (E_p - E_{p'})^2]/(2pp') < y < 1$, leading to the reduced decay width

$$\begin{aligned} \Gamma_{E_p - E_{p'} \leq p + p'} &= \frac{8G_F^2}{(2\pi)^5 E_p} \left(\frac{1}{4} - \sin^2 \theta_W + 2 \sin^4 \theta_W \right) \pi^2 \int \frac{p'^2 dp'}{6E_{p'}} \frac{1}{24pp'} \left\{ 5E_p^6 \right. \\ &\quad - 3(15E_{p'}^2 + 5p^2 - 3p'^2) E_p^4 + 8E_{p'} (10E_{p'}^2 - 6p'^2 + 9pp') E_p^3 \\ &\quad - 3E_p^2 [15E_{p'}^4 - 6(3p^2 - 8p'p + 3p'^2) E_{p'}^2 - (p - p')^2 (5p^2 + 10p'p - 3p'^2)] \\ &\quad - 24E_p E_{p'} p [(2p - 3p') E_{p'}^2 + 3(p - p')^2 p'] + 5E_{p'}^6 + 3E_{p'}^4 (3p^2 - 5p'^2) \\ &\quad \left. - 3E_{p'}^2 (p - p')^2 (3p^2 - 10p'p - 5p'^2) - 5(p - p')^4 (p^2 + 4p'p + p'^2) \right\}. \end{aligned} \quad (4.18)$$

5 Examples

In sections above, we have derived the kinematical threshold and “decay width” of the bremsstrahlung-like process (1.1) for general dispersion relations of neutrino. This was done under the assumptions made in section 2. To check our main results (3.7) and (4.14), we will apply them to muon decay process in subsection 5.1 and to Cohen-Glashow model in subsection 5.2. As further applications, we will use them to study some other models in subsections 5.3, 5.4, 5.5.

5.1 Muon decay

Process (1.1) can be regarded as a three-body decay process by weak interaction. It is analogous to muon decay in standard model. So the basic test of our result (4.14) is comparison with $\mu(p) \rightarrow \nu_\mu(p') + \bar{\nu}_e(k') + e(k)$ by neglecting particle masses in the final states and replacing neutrino with muon in the initial state. Setting $E_{p'} = p'$, $E_p^2 = p^2 + m_\mu^2$, we work out (4.14) directly

$$\Gamma = \frac{G_F^2 m_\mu^6}{192\pi^3 E_p} \left(\frac{1}{4} - \sin^2 \theta_W + 2 \sin^4 \theta_W \right). \quad (5.1)$$

It is different from the muon decay width by a factor $(1/4 - \sin^2 \theta_W + 2 \sin^4 \theta_W)$. This factor should be replaced by 1 if we incorporate the charged current. So our result exactly passes the muon decay test.

5.2 Cohen-Glashow model

The second test is to recover the result in [21]. For this purpose, we set $E_p^2 = p^2(1 + \delta)$, $E_{p'}^2 = p'^2(1 + \delta)$ in (4.14) and get

$$\Gamma = \frac{4}{7} \times \frac{G_F^2 E_p^5 \delta^3}{192\pi^3} \left(\frac{1}{4} - \sin^2 \theta_W + 2 \sin^4 \theta_W \right). \quad (5.2)$$

The decay width of [21] can be numerically reproduced⁵ by taking $\sin^2 \theta_W \simeq 1/4$.

Another main result of this paper is the threshold (3.7). Applying this threshold condition to the model of [21], we find

$$E_{\text{thr.}} = \frac{2m_e + m_\nu}{\sqrt{1 - \frac{1}{1+\delta}}} \simeq \frac{2m_e}{\sqrt{\delta}}, \quad (5.3)$$

the same as the threshold in [21].

5.3 Mass-dependent Lorentz violation

In ref. [23], two of the authors proposed the mass-dependent Lorentz violation scenario to explain the observed neutrino velocity as a function of energy. In this scenario, the mass-energy relation of neutrino has the form

$$1 - v^2 = \lambda - f(\lambda), \quad \lambda = m^2/E^2 \quad (5.4)$$

⁵This corrects the wrong claim in the first version of our manuscript, because the modification of squared amplitude changes the overall factor in the decay width.

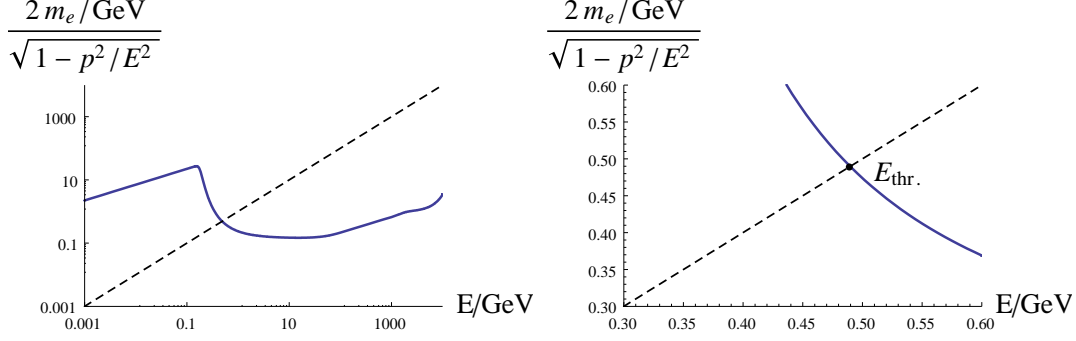


Figure 2. (color online). Numerical solution of the threshold for a toy model of mass-dependent Lorentz violation. The solid blue line depicts function $2m_e(1 - p^2/E_p^2)^{-1/2}$. It crosses the dashed black line at the threshold energy, as highlighted by a black dot in the right graph. The electron/positron mass is set to 0.5 MeV. The toy model and other parameter values are chosen to be the same as in [23].

where $f(\lambda)$ is a model-dependent function. The new function $f(\lambda)$ is useful phenomenologically, because we can get its information directly from experiments which constrain neutrino velocity as a function of energy. With the definition of group velocity $v = dE/dp$, relation (5.4) can be taken as a differential equation and integrated into dispersion relation

$$p = \int_m^E \frac{d\tilde{E}}{\sqrt{1 - \frac{m^2}{\tilde{E}^2} + f\left(\frac{m^2}{\tilde{E}^2}\right)}}. \quad (5.5)$$

A concrete toy model of mass-dependent Lorentz violation was devised in [23], well fitting observational data of neutrino velocity. For the toy model and parameters given in [23], we combine this relation with eq. (3.7), and numerically get the threshold energy at about 0.5 GeV. This is illustrated in figure 2. This threshold is higher than that of the Cohen-Glashow model [21].

Because the dispersion relation takes a very complicated form, it is difficult to work out the decay width in this toy model, even numerically. As an alternative, we will deal with a simplified model in subsection 5.4.

5.4 Velocity of step form

In the toy model of ref. [23], the dependence of velocity on energy looks like a delta function, and well explains the observational data of neutrino velocity. But it is very difficult to work out the decay width (4.14) for that model. As an alternative, let us

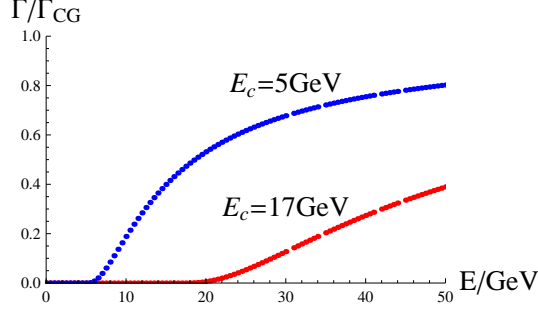


Figure 3. (color online). The dependence of decay width (4.14) on neutrino energy E_p in model (5.6). We have fixed $\delta = 2 \times 10^{-5}$, $E_c = 5$ GeV for the blue points and $E_c = 17$ GeV for the red points.

study a model in which the neutrino's velocity depends on energy in a step form

$$dp/dE = 1 + \left(\frac{1}{\sqrt{1+\delta}} - 1 \right) H(E - E_c). \quad (5.6)$$

Here $H(x)$ is a Heaviside unit step function and E_c is an critical energy.

Fixing $\delta = 2 \times 10^{-5}$, we numerically computed the decay width and got the results in figure 3. From the figure we can see the value of decay width becomes closer and closer to (5.2) as E increases. The figure also tells us that the decay width gets larger if neutrino is faster than light in a wider energy range. Reversing the logic, the decay width can be suppressed by narrowing the energy range in which the neutrino is faster than light. Perhaps this is realizable torturously if the dependence of neutrino velocity on energy takes a comb-like form.

5.5 Horava-Lifshitz model

Motivated by Horava-Lifshitz theories, ref. [26] has studied the dispersion relation for neutrinos of the form

$$E^2 = p^2 + m^2 + \eta' p^2 + \frac{\eta p^4}{M^2}. \quad (5.7)$$

Here the mass of neutrino m is negligible. When the η' correction dominates, this model reduces to the Cohen-Glashow model.

When the η correction dominates, dispersion relation (5.7) becomes

$$E^2 = p^2 + \frac{\eta p^4}{M^2} \quad (5.8)$$

with energy-dependent velocity. For this form of dispersion relation, we can calculate the decay width with (4.14). To the leading order of η , it is

$$\Gamma = \frac{1665}{2002} \times \left(\frac{E_p^2}{M^2} \right)^3 \times \frac{G_F^2 E_p^5 \eta^3}{192\pi^3} \left(\frac{1}{4} - \sin^2 \theta_W + 2 \sin^4 \theta_W \right). \quad (5.9)$$

Remembering that the neutrino velocity is $v^2 - 1 \simeq 3\eta p^2/M^2$, it is convenient to take the notation $\delta = 3\eta E_p^2/M^2$ and rewrite decay width (5.10) as

$$\Gamma = \frac{185}{6006} \times \frac{G_F^2 E_p^5 \delta^3}{192\pi^3} \left(\frac{1}{4} - \sin^2 \theta_W + 2 \sin^4 \theta_W \right). \quad (5.10)$$

At energies relevant to the OPERA experiment, such a decay width is one or two orders smaller than that of the Cohen-Glashow model. This ratio of suppression is consistent with the results of ref. [26]. A naive application of (3.7) to (5.8) yields $\eta p_{\text{thr.}}^4/M^2 = (2m_e + m_\nu)^2$, which gives the threshold energy in leading order

$$E_{\text{thr.}} \simeq \frac{\sqrt{2m_e M}}{\eta^{1/4}}. \quad (5.11)$$

It is unsafe to take $E_{\text{thr.}} \simeq 2m_e \sqrt{3/\delta}$, because here δ is energy-dependent.

6 Conclusion

In this paper, under the assumptions enumerated in section 2, we studied the kinematic threshold and decay width of superluminal neutrinos for the bremsstrahlung-like process (1.1). This was done for general dispersion relations of neutrino, without resorting to any nontrivial frame such as the effective “rest frame”. The main results are represented by eqs. (3.7) and (4.14). Our results confirmed and generalized the previous results in [21, 22].

Before concluding this paper, we would like to make some relevant remarks on the assumption of squared amplitude, which leaves a loose end for future investigation. As has been emphasized in section 2, when calculating the decay width, we assumed that the squared amplitude is the same as that in standard model. Strictly speaking, this is not always a consistent assumption when general dispersion relations are involved. In general, dispersion relations will enter into both kinematics and dynamics of particle physics. However, without an assumption on the squared amplitude, we cannot do any calculation about decay width. At the same time, since the deviation of neutrino dispersion relation is not too far from special relativity, we expect that the deformed amplitude should not deviate significantly from the standard model. Therefore, our assumption (4.2) is not only necessary but also natural to some extent. We thus expect our result provides a good estimation of decay width in order of magnitude. Of course,

further investigation is required to confirm this expectation and improve the present situation. See ref. [27] for a recent progress along this direction.

Another interesting project is employing our general results to rule out more phenomenological models and hunt for viable models, as shown by some examples in section 5. We feel this project will be challenging but rewarding, given the importance of special relativity in modern physics.

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